

Analogues of Jacobi's derivative formula

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Abstract In this paper, we obtain analogues of Jacobi's derivative formula in terms of the theta constants with rational characteristics. For this purpose, we use the arithmetic formulas of the number of representations of a natural number n , ($n = 1, 2, \dots$) as the sum of two squares, or the sum of a square and twice a square, which is given by

$$S_2(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid n = x^2 + y^2\} = 4 \sum_{d \mid n, d: \text{odd}} (-1)^{\frac{d-1}{2}},$$

$$S_{1,2}(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid n = x^2 + 2y^2\} = 2(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)),$$

where for the positive integers j, k, n , $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \pmod{k}$.

Key Words: theta functions; rational characteristics; Jacobi's derivative formula; the sum of two squares.

MSC(2010) 14K25; 11E25

1 Introduction

Throughout this paper, set $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. \mathbb{Z} denotes the set of rational integers. Furthermore, for the positive integers j, k, n , $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \pmod{k}$.

Following Farkas and Kra [6], we introduce the theta function with characteristics, which is defined by

$$\theta \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] (\zeta, \tau) := \sum_{n \in \mathbb{Z}} \exp \left(2\pi i \left[\frac{1}{2} \left(n + \frac{\epsilon}{2} \right)^2 \tau + \left(n + \frac{\epsilon}{2} \right) \left(\zeta + \frac{\epsilon'}{2} \right) \right] \right),$$

where $\epsilon, \epsilon' \in \mathbb{R}$, $\zeta \in \mathbb{C}$, and $\tau \in \mathbb{H}^2$, the upper half plane. The theta constants are given by

$$\theta \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] := \theta \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] (0, \tau), \quad \theta' \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] := \frac{\partial}{\partial \zeta} \theta \left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right] (\zeta, \tau) \Big|_{\zeta=0}.$$

Farkas and Kra [6] treated the theta constants with rational characteristics, that is, the case where ϵ and ϵ' are both rational numbers, and derived a number of interesting theta constant identities.

Our concern is with Jacobi's derivative formula, which is given by

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\pi \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (1.1)$$

which implies that for $q \in \mathbb{C}$ with $|q| < 1$,

$$\prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

This is equivalent to the following identity:

$$\eta^3(\tau) = \sum_{n=0}^{\infty} \left(\frac{-1}{n} \right) n \exp \left(\frac{2\pi i n^2 \tau}{8} \right),$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = \exp(2\pi i \tau)$, and

$$\left(\frac{-1}{n} \right) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

For generalizations of this derivative formula to a higher genus, see Igusa [3].

Farkas [7] derived the following theta constant identity:

$$\begin{aligned} & \frac{6\theta' \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} (0, \tau)}{\zeta_6 \theta^3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} (0, \tau) + \theta^3 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0, \tau) + \zeta_6^5 \theta^3 \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} (0, \tau)} \\ &= \frac{2\pi i x^{\frac{1}{12}}}{\prod_{n=0}^{\infty} (1 - x^{3n+1})(1 - x^{3n+2})} = 2\pi i \frac{e^{\frac{\pi i}{6}} \theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} (0, \tau)}{\sqrt{3} \theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0, 3\tau)} = 2\pi i \frac{\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0, 9\tau)}{\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0, 3\tau)}, \end{aligned} \quad (1.2)$$

where $x = \exp(2\pi i \tau)$ and $\zeta_6 = \exp(2\pi i/6)$. The identity (1.2) can be viewed as an analogue of Jacobi's derivative formula.

In this paper, we express $\theta' \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \theta' \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\theta' \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$ by the theta constants with rational characteristics. For this purpose, we note that

$$\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) = \left(\sum_{n \in \mathbb{Z}} x^{n^2} \right)^2 = 1 + \sum_{n=1}^{\infty} S_2(n) x^n, \quad x = \exp(\pi i \tau), \quad (1.3)$$

and

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) = 1 + \sum_{n=1}^{\infty} S_{1,2}(n) x^n, \quad x = \exp(\pi i \tau), \quad (1.4)$$

where

$$S_2(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid n = x^2 + y^2\} = 4 \sum_{d \mid n, d: \text{odd}} (-1)^{\frac{d-1}{2}}, \quad (1.5)$$

and

$$S_{1,2}(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid n = x^2 + 2y^2\} = 2(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)). \quad (1.6)$$

For the proof of equations (1.5) and (1.6), see Berndt [1, pp. 56, 74]. For the elementary proof, see Dickson [2, pp.68].

Our main theorems are as follows:

Theorem 1.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta' \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau) = -\pi \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau), \quad (1.7)$$

which implies that

$$\frac{\eta^9(2\tau)}{\eta^3(\tau)\eta^3(4\tau)} = \sum_{n=0}^{\infty} \left(\frac{-2}{n} \right) n \exp \left(\frac{2\pi i n^2 \tau}{8} \right), \quad (1.8)$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = \exp(2\pi i \tau)$, and

$$\left(\frac{-2}{n} \right) = \begin{cases} +1, & \text{if } n \equiv 1 \text{ or } 3 \pmod{8}, \\ -1, & \text{if } n \equiv 5 \text{ or } 7 \pmod{8}, \\ 0, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

Theorem 1.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) = -\pi \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \left\{ \sqrt{2} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) - \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \right\},$$

and

$$\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) = -\pi \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \left\{ \sqrt{2} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \right\}.$$

Equation (1.8) was proved by Zucker [10], [11] and Köhler [4]. Noted is that they did not use theta functions with rational characteristics. Therefore, we believe that equation (1.7) is new. For the theory of the eta products and theta series identities, see Köhler [5].

This paper is organized as follows. In Section 2, we review the properties of the theta functions. In Sections 3 and 4, we prove Theorems 1.1 and 1.2. In Section 3, we especially note that equation (1.7) has a relationship with a certain kind of partition number. In Section 5, we derive more theta constant identities.

Acknowledgments

We are grateful to Professor Nakayashiki and Professor Nishiyama for their useful comments. We also thank the referee for recommending various improvements to the paper.

2 The properties of the theta functions

2.1 Basic properties

We first note that for $m, n \in \mathbb{Z}$,

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta + n + m\tau, \tau) = \exp(2\pi i) \left[\frac{n\epsilon - m\epsilon'}{2} - mz - \frac{m^2\tau}{2} \right] \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau), \quad (2.1)$$

and

$$\theta \begin{bmatrix} \epsilon + 2m \\ \epsilon' + 2n \end{bmatrix} (\zeta, \tau) = \exp(\pi i \epsilon n) \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau). \quad (2.2)$$

Furthermore,

$$\theta \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (\zeta, \tau) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\zeta, \tau) \text{ and } \theta' \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (\zeta, \tau) = -\theta' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\zeta, \tau).$$

For $m, n \in \mathbb{R}$, we see that

$$\begin{aligned} & \theta \left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] \left(\zeta + \frac{n + m\tau}{2}, \tau \right) \\ &= \exp(2\pi i) \left[-\frac{mz}{2} - \frac{m^2\tau}{8} - \frac{m(\epsilon' + n)}{4} \right] \theta \left[\begin{matrix} \epsilon + m \\ \epsilon' + n \end{matrix} \right] (\zeta, \tau). \end{aligned} \quad (2.3)$$

We note that $\theta \left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] (\zeta, \tau)$ has only one zero in the fundamental parallelogram, which is given by $\zeta = \frac{1-\epsilon}{2}\tau + \frac{1-\epsilon'}{2}$.

2.2 Jacobi's triple product identity

All the theta functions have infinite product expansions, which are given by

$$\begin{aligned} \theta \left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] (\zeta, \tau) &= \exp \left(\frac{\pi i \epsilon \epsilon'}{2} \right) x^{\frac{\epsilon^2}{4}} z^{\frac{\epsilon}{2}} \\ &\prod_{n=1}^{\infty} (1 - x^{2n}) (1 + e^{\pi i \epsilon'} x^{2n-1+\epsilon} z) (1 + e^{-\pi i \epsilon'} x^{2n-1-\epsilon} / z), \end{aligned} \quad (2.4)$$

where $x = \exp(\pi i \tau)$ and $z = \exp(2\pi i \zeta)$.

2.3 Spaces of N -th order θ -functions

Following Farkas and Kra [6], we define $\mathcal{F}_N \left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right]$ to be the set of entire functions f that satisfy the two functional equations,

$$f(\zeta + 1) = \exp(\pi i \epsilon) f(\zeta),$$

and

$$f(\zeta + \tau) = \exp(-\pi i)[\epsilon' + 2N\zeta + N\tau] f(\zeta), \quad \zeta \in \mathbb{C}, \tau \in \mathbb{H}^2,$$

where N is a positive integer and $\left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] \in \mathbb{R}^2$. This set of functions is called the space of N -th order θ -functions with characteristic $\left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right]$. Note that

$$\dim \mathcal{F}_N \left[\begin{matrix} \epsilon \\ \epsilon' \end{matrix} \right] = N.$$

For its proof, see Farkas and Kra [6, pp.133].

2.4 Lemma of Farkas and Kra

We recall the lemma of Farkas and Kra [6, pp.78].

Lemma 2.1. *For all characteristics $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \begin{bmatrix} \delta \\ \delta' \end{bmatrix}$ and all $\tau \in \mathbb{H}^2$, we have*

$$\begin{aligned} & \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \delta \\ \delta' \end{bmatrix} (0, \tau) \\ &= \theta \begin{bmatrix} \frac{\epsilon+\delta}{2} \\ \epsilon' + \delta' \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} \frac{\epsilon-\delta}{2} \\ \epsilon' - \delta' \end{bmatrix} (0, 2\tau) + \theta \begin{bmatrix} \frac{\epsilon+\delta}{2} + 1 \\ \epsilon' + \delta' \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} \frac{\epsilon-\delta}{2} + 1 \\ \epsilon' - \delta' \end{bmatrix} (0, 2\tau). \end{aligned}$$

3 On Theorem 1.1

3.1 Proof of equation (1.7)

Using equation (1.5), we prove Theorem 1.1.

Proof. The Jacobi triple product identity (2.4) yields

$$\begin{aligned}
\frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \right] (0, \tau)} &= \frac{d}{d\zeta} \log \theta \left[\begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \right] (\zeta, \tau) \Big|_{\zeta=0} \\
&= \frac{dz}{d\zeta} \cdot \frac{d}{dz} \log \left(e^{\frac{\pi i}{4}} x^{1/4} z^{1/2} \prod_{n=1}^{\infty} (1 - x^{2n})(1 + ix^{2n}z)(1 - i \frac{x^{2n-2}}{z}) \right) \Big|_{\zeta=0} \\
&= 2\pi i z \left\{ \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{ix^{2n}}{1 + ix^{2n}z} + \sum_{n=1}^{\infty} \frac{-ix^{2n-2}(-z^{-2})}{1 - ix^{2n-2}z^{-1}} \right\} \Big|_{\zeta=0} \\
&= 2\pi i \left\{ \frac{1}{2} + \frac{i}{1-i} + \sum_{n=1}^{\infty} \frac{ix^{2n}}{1 + ix^{2n}} + \sum_{n=1}^{\infty} \frac{ix^{2n}}{1 - ix^{2n}} \right\} \\
&= 2\pi i \left\{ \frac{i}{2} + 2i \sum_{n=1}^{\infty} \frac{x^{2n}}{1 + x^{4n}} \right\} \\
&= -\pi \left\{ 1 + 4 \sum_{n=1}^{\infty} x^{2n} \sum_{m=0}^{\infty} (-1)^m x^{4nm} \right\} \\
&= -\pi \left\{ 1 + 4 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m x^{2n(2m+1)} \right\} \\
&= -\pi \left\{ 1 + 4 \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N, d: \text{odd}} (-1)^{\frac{d-1}{2}} \right) \right\} \\
&= -\pi \sum_{n=0}^{\infty} S_2(n) x^{2n} \\
&= -\pi \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau),
\end{aligned}$$

where $x = \exp(\pi i \tau)$ and $z = \exp(2\pi i \zeta)$. □

3.2 Proof of equation (1.8)

Proof. Theorem 1.1 and Jacobi's triple product identity (2.4) imply that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^9}{(1 - q^n)^3 (1 - q^{4n})^3} = \sum_{n=0}^{\infty} \left(\frac{-2}{n} \right) n q^{t_{(n-1)/2}}, \quad (3.1)$$

where for $n \in \mathbb{Z}$, $t_n = n(n+1)/2$.

Replacing q by q^8 and multiplying both sides of equation (3.1) by q , we obtain

$$q \prod_{n=1}^{\infty} \frac{(1 - q^{16n})^9}{(1 - q^{8n})^3 (1 - q^{32n})^3} = \sum_{n=0}^{\infty} \left(\frac{-2}{n} \right) n q^{n^2}.$$

Setting $q = \exp(2\pi i \tau)$, we obtain

$$\frac{\eta^9(16\tau)}{\eta^3(8\tau)\eta^3(32\tau)} = \sum_{n=0}^{\infty} \left(\frac{-2}{n} \right) n \exp(2\pi i n^2 \tau).$$

Replacing τ by $\tau/8$, we can prove equation (1.8). □

4 Proof of Theorem 1.2

4.1 Applications of $S_2(n)$ and $S_{1,2}(n)$

Proposition 4.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)} - \frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)} = 2\pi\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau), \quad (4.1)$$

and

$$\frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)} + \frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)} = -2\sqrt{2}\pi\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau). \quad (4.2)$$

Proof. Jacobi's triple product identity (2.4) yields

$$\begin{aligned} \frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{1}{4} \end{smallmatrix} \right] (0, \tau)} &= 2\pi i \left\{ \frac{i}{\sqrt{2}} - \frac{i}{2} + 2i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} \sin \frac{\pi d}{4} \right) \right\}, \\ \frac{\theta' \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)}{\theta \left[\begin{smallmatrix} 1 \\ \frac{3}{4} \end{smallmatrix} \right] (0, \tau)} &= 2\pi i \left\{ \frac{i}{\sqrt{2}} + \frac{i}{2} + 2i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} \sin \frac{3\pi d}{4} \right) \right\}, \end{aligned}$$

where $x = \exp(\pi i \tau)$.

We first treat equation (4.1). For this purpose, we have

$$\begin{aligned}
\frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} - \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} &= 2\pi i \left\{ -i + 2i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} \left[\sin \frac{\pi d}{4} - \sin \frac{3\pi d}{4} \right] \right) \right\} \\
&= 2\pi i \left\{ -i - 4i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} \cos \frac{\pi d}{2} \sin \frac{\pi d}{4} \right) \right\} \\
&= 2\pi i \left\{ -i - 4i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N, d \equiv 2 \pmod{8}} 1 - \sum_{d|N, d \equiv 6 \pmod{8}} 1 \right) \right\} \\
&= 2\pi \left\{ 1 + 4 \sum_{N=1}^{\infty} x^{4N} \left(\sum_{d|N, d: \text{odd}} (-1)^{\frac{d-1}{2}} \right) \right\} \\
&= 2\pi \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau).
\end{aligned}$$

We next deal with (4.2). For this purpose, we have

$$\begin{aligned}
\frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} + \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} &= 2\pi i \left\{ \sqrt{2}i + 2i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} \left[\sin \frac{\pi d}{4} + \sin \frac{3\pi d}{4} \right] \right) \right\} \\
&= 2\pi i \left\{ \sqrt{2}i + 2i \sum_{N=1}^{\infty} x^{2N} \left(\sum_{d|N} (-1)^{d-1} 2 \sin \frac{\pi d}{2} \cos \frac{\pi d}{4} \right) \right\} \\
&= 2\pi i \left\{ \sqrt{2}i + 2i \sum_{N=1}^{\infty} x^{2N} \sqrt{2} (d_{1,8}(N) + d_{3,8}(N) - d_{5,8}(N) - d_{7,8}(N)) \right\} \\
&= -2\sqrt{2}\pi \left\{ 1 + 2 \sum_{N=1}^{\infty} S_{1,2}(N) x^{2N} \right\} \\
&= -2\sqrt{2}\pi \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau).
\end{aligned}$$

□

4.2 Proof of Theorem 1.2

Proof. The theorem follows from Proposition 4.1. \square

4.3 Another expressions of the derivative formulas

Proposition 4.2. *For every $(\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2$, we have*

$$\begin{aligned} & \frac{\theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) + \zeta_8^6 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix} (\zeta, \tau)}{\theta \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (\zeta, \tau)} \\ &= - \frac{8\zeta_8^3 \left(\theta^3 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^3 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \right)}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau)}, \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} & \frac{\theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) + \zeta_8^2 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{7}{4} \end{bmatrix} (\zeta, \tau)}{\theta \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} (\zeta, \tau)} \\ &= - \frac{8\zeta_8 \left(\theta^3 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^3 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \right)}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau)}, \end{aligned} \quad (4.4)$$

where $\zeta_8 = \exp(2\pi i/8)$.

Proof. We treat equation (4.3). Equation (4.4) can be proved in the same way. For this purpose, we set

$$f(\zeta) = \frac{\theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) + \zeta_8^6 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau) - \zeta_8 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix} (\zeta, \tau)}{\theta \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (\zeta, \tau)}.$$

It can be easily verified that $f(\zeta)$ is an elliptic function with no poles, because both the numerator and denominator of $f(\zeta)$ have the same zeros. Thus, $f(\zeta)$ is constant.

If we differentiate both the numerator and denominator of $f(\zeta)$ and set $\zeta = \frac{3\tau}{8}$, we obtain equation (4.3). \square

Setting $\zeta = 0$ in Proposition 4.2 and using Jacobi's derivative's formula (1.1), we obtain the following corollary:

Corollary 4.3. *For every $\tau \in \mathbb{H}^2$, we have*

$$\begin{aligned}
& \theta^3 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} - \theta^3 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \\
&= \frac{\pi}{8\zeta_8^3} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \frac{\theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} + \zeta_8^6 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} - \zeta_8 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix}} \\
&= \frac{\pi}{8\zeta_8} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \frac{\theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} - \zeta_8 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} + \zeta_8^2 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix} - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{3}{4} \\ \frac{7}{4} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} \theta \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} \theta \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix}},
\end{aligned}$$

where $\zeta_8 = \exp(2\pi i/8)$.

By Lemma 2.1 and Corollary 4.3, we obtain the following proposition:

Proposition 4.4. *For every $\tau \in \mathbb{H}^2$, we have*

$$\begin{aligned}
& \theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \cdot \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \cdot \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \\
&= -2\pi\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\}.
\end{aligned}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
\theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (0, \tau) &= \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \zeta_8^5 \theta \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau), \\
\theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (0, \tau) &= \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \zeta_8^5 \theta \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau), \\
\theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (0, \tau) &= \zeta_8 \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \zeta_8^2 \theta \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau), \\
\theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix} (0, \tau) &= \zeta_8 \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \zeta_8^2 \theta \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau).
\end{aligned}$$

Furthermore, by Lemma 2.1, we obtain

$$\begin{aligned}
& \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \zeta_8^3 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (0, \tau) + \zeta_8^6 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (0, \tau) - \zeta_8 \theta^4 \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix} (0, \tau) \\
&= 2\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \left\{ \theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) - \zeta_8^3 \theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \right\} \\
&\quad - 2\zeta_8^2 \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau) \left\{ \zeta_8^3 \theta^2 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (0, 2\tau) - \theta^2 \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} (0, 2\tau) \right\} \\
&= 4\theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
& \theta \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix} (0, \tau) \\
&= \zeta_8 \theta \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) - \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\}, \\
& \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau) \\
&= 4\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \theta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) - \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\},
\end{aligned}$$

which proves the proposition. \square

Theorem 4.5. *For every $\tau \in \mathbb{H}^2$, we have*

$$\begin{aligned}
\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) &= \frac{-2\pi\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau)}{\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \times \\
&\times \left(\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\} + \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \right),
\end{aligned}$$

and

$$\begin{aligned}
\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) &= \frac{-2\pi\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau)}{\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \times \\
&\times \left(\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\} + \theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \right).
\end{aligned}$$

Proof. We first note that

$$\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) = \theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \left(1 - \left\{ \frac{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} \right\}^4 \right).$$

Taking the limit $\tau \rightarrow i\infty$, we have

$$\begin{aligned} 1 - \left\{ \frac{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, i\infty)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, i\infty)} \right\}^4 &= 1 - \left(\frac{\cos^2 \frac{3\pi}{8}}{\cos^2 \frac{\pi}{8}} \right)^2 \\ &= 12\sqrt{2} - 16 \neq 0, \end{aligned}$$

which implies that

$$\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \neq 0.$$

Considering equation (4.1) and Proposition 4.4, we obtain the theorem. \square

Theorem 4.6. *For every $\tau \in \mathbb{H}^2$, we have*

$$\begin{aligned} \theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) &= \frac{-\pi \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau)}{\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) + \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \times \\ &\times \left(\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\} \right. \\ &\quad \left. + 2\sqrt{2}\theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \right), \end{aligned}$$

and

$$\begin{aligned} \theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) &= \frac{\pi \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau)}{\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) + \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \times \\ &\times \left(\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 4\tau) + 3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 4\tau) \right\} \right. \\ &\quad \left. - 2\sqrt{2}\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \right). \end{aligned}$$

Proof. We first note that

$$\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) + \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) = \theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) \left(1 + \left\{ \frac{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} \right\}^4 \right).$$

Taking the limit $\tau \rightarrow i\infty$, we have

$$\begin{aligned} 1 + \left\{ \frac{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, i\infty)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, i\infty)} \right\}^4 &= 1 + \left(\frac{\cos^2 \frac{3\pi}{8}}{\cos^2 \frac{\pi}{8}} \right)^2 \\ &= 12\sqrt{2} + 18 \neq 0, \end{aligned}$$

which implies that

$$\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau) + \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau) \neq 0.$$

By Lemma 2.1, we next note that

$$\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) = 2\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau).$$

Considering equation (4.2) and Proposition 4.4, we obtain the theorem. \square

5 More theta constant identities

Theorem 5.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} = 0, \quad (5.1)$$

$$\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = 0, \quad (5.2)$$

$$\theta^4 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} - \theta^4 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \quad (5.3)$$

Proof. We first prove equation (5.1). Equations (5.2) and (5.3) can be proved in the same way.

For this purpose, we note that $\dim \mathcal{F}_2 \begin{bmatrix} \frac{2}{4} \\ 0 \end{bmatrix} = 2$ and

$$\begin{aligned} & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau), \quad \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} (\zeta, \tau), \\ & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau), \quad \theta^2 \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \in \mathcal{F}_2 \begin{bmatrix} \frac{2}{4} \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, there exists complex numbers x_1, x_2, x_3, x_4 , not all zero such that

$$\begin{aligned} & x_1 \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) + x_2 \theta \begin{bmatrix} \frac{1}{4} \\ \frac{2}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} (\zeta, \tau) \\ & + x_3 \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau) + x_4 \theta^2 \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) = 0. \end{aligned}$$

Substituting

$$\zeta = \frac{3\tau \pm 3}{8}, \frac{3\tau \pm 2}{8}, \frac{3\tau \pm 1}{8}, \frac{3\tau}{8},$$

we obtain

$$A\mathbf{x} = \mathbf{0}, \quad (5.4)$$

where

$$A = \begin{pmatrix} 0 & \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} & \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} & \theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \\ -\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} & 0 & \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} & \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \\ -\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} & -\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} & 0 & \theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \\ -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} & -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} & -\theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} & 0 \end{pmatrix},$$

and $\mathbf{x} = {}^t(x_1, x_2, x_3, x_4)$.

Since a system of equations (5.4) has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$, it follows that $\det A = 0$, which proves equation (5.1).

Equations (5.2) and (5.3) can be proved by considering that

$$\begin{aligned} & \theta^2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (\zeta, \tau), & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{2}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} (\zeta, \tau), \\ & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau), & \theta^2 \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \in \mathcal{F}_2 \begin{bmatrix} \frac{2}{4} \\ 0 \end{bmatrix}, \end{aligned}$$

and that

$$\begin{aligned} & \theta^2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (\zeta, \tau), & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{7}{4} \end{bmatrix} (\zeta, \tau), \\ & \theta \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix} (\zeta, \tau), & \theta^2 \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} (\zeta, \tau) \in \mathcal{F}_2 \begin{bmatrix} \frac{2}{4} \\ 0 \end{bmatrix}. \end{aligned}$$

□

Remark

Since matrix A is a skew-symmetric matrix, it follows that equation (5.1) is expressed in terms of a Pfaffian. In the same way, equations (5.2) and (5.3) can be given in terms of Pfaffian. We note that in [9] we derived some theta constant identities by considering the determinant structure of a matrix of the theta constants.

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